# UK INTERMEDIATE MATHEMATICAL CHALLENGE

**February 7th 2013**

# EXTENDED SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended (the shorter solutions have also been appended to the end of this document). In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Intermediate Mathematical Challenge (IMC) is a multiple choice contest, in which you are presented with five alternative answers, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC, and we often give first a solution using this approach.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So usually we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the Intermediate Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to enquiry@ukmt.co.uk

or by post to IMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

**Quick Marking Guide**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D | B | E | D | E | E | A | D | A | C | C | D | B | D | B | A | D | C | E | B | B | D | A | B | D |

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1. Which of the following is divisible by 6?

A one million minus one  B one million minus two  C one million minus three

D one million minus four  E one million minus five

*Solution:* **D**

It helps to first write the options we are given in standard form using digits:

A 999 999  B 999 998  C 999 997  D 999 996  E 999 995

An integer is divisible by 6 if and only if it is divisible by 2 and by 3. It is easy to see that, of the given options, only 999 998 and 999 996 are divisible by 2, and, of these, only 999 996 is divisible by 3.

**Extension Problem**

1.1 Because 6 and 9 are both multiples of 3 it is easy to see that $999\,996 = 3 \times 333\,332$ and hence that 999 996 is a multiple of 3. In other cases the following test is useful:

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

For example, consider the number 528 337 824. The sum of its digits is $5+2+8+3+3+7+8+2+4=42$ and the sum of the digits of 42 is $4+2=6$. As 6 is divisible by 3, by the above test, 42 is also divisible by 3 and hence, using the test again, we deduce that 528 337 824 is divisible by 3.

Use the test to determine which of the following numbers is divisible by 3:

a) 147,  b) 2 947,  c) 472 286,  d) 824 635 782  e) 123 456 789.

1.2 Give an argument to prove that the above test for divisibility by 3 is correct.

1.3 Find a similar test for divisibility by 9.

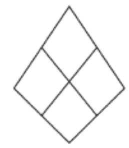
2. A machine cracks open 180 000 eggs per hour. How many eggs is that per second?

A 5  B 50  C 500  D 5000  E 50 000

*Solution:* **B**

There are 60 seconds in a minute and 60 minutes in an hour. So there are $60 \times 60 = 3600$ seconds in an hour. So 180 000 eggs per hour is the same as $\frac{180\,000}{3\,600} = \frac{1800}{36} = 50$ eggs per second.

3. How many quadrilaterals are there is this diagram, which is constructed using 6 straight lines?



A 4 B 5 C 7 D 8 E 9

*Solution:* **E**

In a question of this type we need to try and find a systematic approach so that we count all the quadrilaterals but do not count any of them twice. The method we use (there are others) is to count all the quadrilaterals according to how many of the four smaller quadrilaterals shown in the diagram make them up.

We see that there are 4 quadrilaterals each made up of just one of the small quadrilaterals, 4 made up of two of the smaller quadrilaterals, and 1 made up of all four of the smaller quadrilaterals. This makes a total of $4+4+1=9$ quadrilaterals altogether.

These are shown in the diagram below.

4. A standard pack of pumpkin seeds contains 40 seeds. A special pack contains 25% more seeds. Rachel bought a special pack and 70% of the seeds germinated. How many pumpkin plants did Rachel have?

A 20 B 25 C 28 D 35 E 50

*Solution:* **D**

As a special pack contains 25% more seeds than a standard pack of 20, it contains $\frac{125}{100}\times 40=\frac{5}{4}\times 40=50$ seeds. As 70% of these germinated, $\frac{70}{100}\times 50=35$ seeds germinated. So Rachel had 35 pumpkin plants.

5. The northern wheatear is a small bird weighing less than an ounce. Some northern wheatears migrate from sub-Saharan Africa to their Arctic breeding grounds, travelling almost 15 000 km. The journey takes just over 7 weeks. Roughly how far do they travel each day, on average?

A 1 km B 9 km C 30 km D 90 km E 300 km

*Solution:* **E**

There are $7\times 7=49$ days in 7 weeks. So we will get a good estimate of the average distance travelled by a wheatear in 7 weeks by dividing the total distance travelled by 50. Now $\frac{15\,000}{50}=\frac{1\,500}{5}=300$. So the average distance travelled is roughly 300 km.

6. Which of the following has the least value?

A $1^0 - 0^1$ B $2^1 - 1^2$ C $3^2 - 2^3$ D $4^3 - 3^4$ E $5^4 - 4^5$

*Solution:* **E**

*Solution:* **E**

The only feasible method here is to evaluate each of the options. We see that

$1^0 - 0^1 = 1 - 0 = 1$, $2^1 - 1^2 = 2 - 1 = 1$, $3^2 - 2^3 = 9 - 8 = 1$, $4^3 - 3^4 = 64 - 81 = -17$ and $5^4 - 4^5 = 625 - 1024 = -399$. Of these, $5^4 - 4^5$ is the least.

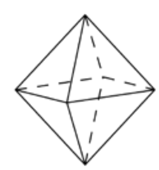
**Extension Problems**

6.1 The given options are the first 5 terms in the sequence whose general term is $n^{n-1} - (n-1)^n$. Notice how misleading the first three terms of this sequence are. They are all equal to 1, but if you had "spotted the pattern" and so thought that *all* the terms of the sequence were equal to 1, you would have made a big mistake!

Investigate what happens to the terms of this sequence as *n* get larger and larger.

6.2 Find the least value of *n* for which $n^{n-1} - (n-1)^n < -1\,000\,000$.

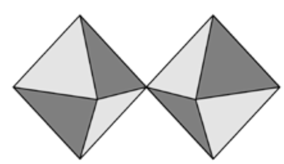
7. The faces of a regular octahedron are to be painted so that no two faces which have an edge in common are painted in the same colour. What is the smallest number of colours required?



A 2 B 3 C 4 D 6 E 8

*Solution:* **A**

Clearly, more than one colour is needed! On the other hand it is possible to paint the faces using just two colours so that faces with an edge in common are painted in different colours. This is done as shown in the diagram.



back view front view

**Extension Problems**

7.1 Consider the same problem for the other regular polyhedrons, that is, the cube, tetrahedron, dodecahedron and icosahedron. For each of these, determine the smallest number of colour needed to paint its faces so that faces which have an edge in common are painted different colours.

7.2 Now consider other, non-regular polyhedrons, and consider the same problem for them.

7.3 Which polyhedrons are like the octahedron in the sense that only two colours are needed? Make a conjecture about a test for which polyhedrons need only two colours. Then try to prove that your conjecture is correct.

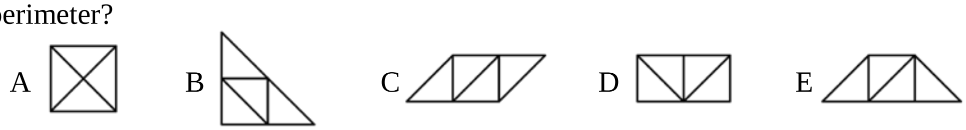
8. Jim rolled some dice and was surprised that the sum of the scores on the dice was equal to the product of the scores on the dice. One of the dice showed a score of 2, one showed 3 and one showed 5. The rest showed a score of 1. How many dice did Jim roll?

A 10 B 13 C 17 D 23 E 30

*Solution:* **D**

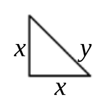
Suppose there were $n$ dice which showed a score of 1. Then the sum of the scores on the dice was $n \times 1 + 2 + 3 + 5 = n + 10$. The product of the scores was $1^n \times 2 \times 3 \times 5 = 1 \times 2 \times 3 \times 5 = 30$. Therefore, $n + 10 = 30$ and hence $n = 20$. So, including the three dice which showed the scores of 2, 3 and 5, there were 23 dice altogether.

9. Jane has 20 identical cards in the shape of an isosceles right-angled triangle. She uses the cards to make the five shapes below. Which of the shapes has the shortest perimeter?




*Solution:* **A**

Let the side lengths of the isosceles right-angled triangle be $x$, $x$ and $y$ as shown in the diagram. By Pythagoras' Theorem $y^2 = x^2 + x^2 = 2x^2$ and hence $y = \sqrt{2}x$. We then see that the perimeters of the shapes are as follows:




A $4\sqrt{2}x$, B $(4+2\sqrt{2})x$, C $(4+2\sqrt{2})x$, D 6 and E $(4+2\sqrt{2})x$.

Now , as $1 < \sqrt{2} < 1.5$, we have $4\sqrt{2} < 4 \times 1.5 = 6 = 4 + 2 < 4 + 2\sqrt{2}$. So shape A has the shortest perimeter.

[Note: that, as the value of $x$ is irrelevant, we could have, for example, put $x = 1$ to begin with.]

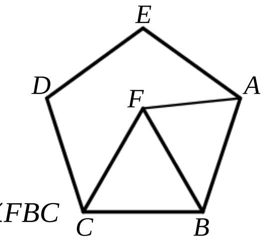
**Extension Problems**

9.1 In the above solution we have used the fact that $1 < \sqrt{2} < 1.5$. If you know that the decimal expansion of $\sqrt{2}$ is 1.414213…, the truth of these inequalities is obvious. Suppose, however, that all you know about $\sqrt{2}$ is that it is the positive number which is the solution of the equation $x^2 = 2$. How can you deduce from this that $1 < \sqrt{2} < 1.5$?

9.2 Use the same method to prove that $\frac{24}{17} < \sqrt{2} < \frac{25}{17}$.

| | |
|---|---|
| 10. | *ABCDE* is a regular pentagon and *BCF* is an equilateral triangle such that *F* is inside *ABCDE*. What is the size of $\angle FAB$? |
| | A $48^\circ$    B $63^\circ$    C $66^\circ$    D $69^\circ$    E $72^\circ$ |

*Solution:* **C**




Since the pentagon *ABCDE* is regular, $BC = AB$. Since the triangle *BCF* is equilateral, $BC = FB$. Therefore, $AB = FB$. So the triangle *ABF* is isosceles, and hence $\angle FAB = \angle BFA$.

$\angle ABC = 108^\circ$, as it is the interior angle of a regular pentagon. As the triangle *BCF* is equilateral, $\angle FBC = 60^\circ$. Therefore $\angle ABF = \angle ABC - \angle FBC$ $= 108^\circ - 60^\circ = 48^\circ$. So, from triangle *ABF*, we can deduce that $\angle FAB + \angle BFA$ $= 180^\circ - 48^\circ = 132^\circ$. Since $\angle FAB = \angle BFA$, it follows that $\angle FAB = \frac{1}{2}(132^\circ) = 66^\circ$.

**Extension Problems**

10.1 In this solution we have taken it for granted that the interior angle of a regular pentagon is $108^\circ$. Explain why this is correct.

10.2 What is the interior angle of a regular dodecagon (a 12 sided figure)?

10.3 Suppose that the interior angle of a given regular polygon is $162^\circ$. How many vertices does this polygon have?

| | |
|---|---|
| 11. | For which of the following numbers is the sum of all its factors *not* equal to a square number? |
| | A 3    B 22    C 40    D 66    E 70 |

*Solution:* **C**

The most straightforward method here is to just list all the factors of the numbers, add them up, and see in which cases this sum is not a square number

| *number* | *factors* | *sum of factors* |
|---|---|---|
| 3 | 1, 3 | 4 |
| 22 | 1, 2, 11, 22 | 36 |
| 40 | 1, 2, 4, 5, 8, 10, 20, 40 | 90 |

Since 90 is not a square, in the context of the IMC, you could stop here. However, for completeness, we also calculate the sum of the factors of the other two numbers we are given.

| | | |
|---|---|---|
| 66 | 1, 2, 3, 6, 11, 22, 33, 66 | 144 |
| 70 | 1, 2, 5, 7, 10, 14, 35, 70 | 144 |

So, for each option given, other than 40, the sum of its factors is a square number.

**Extension Problems**

When a number is quite small, listing all its factors and then adding them up is as good a way as any for calculating the sum of its factors. In problems 11.1 to 11.7 we ask you to think about a formula for working out the sum of the factors of a number from the way it can be factorized into prime numbers. Problem 11.8 asks about a similar formula for the *number* of different factors of a number.

11.1 We begin with the easiest case of a prime number which has only 1 and itself as factors. For example, the sum of the factors of 2 is $1+2=3$. In general, the factors of a prime number *p* are 1 and *p*, and so the sum of its factors is $1+p$. Next we consider the products of two prime numbers. For example, consider $22=2\times 11$. How can you relate this factorization of 22, to the sum of its factors, which we saw was 36. Does the same thing work for 15 and 35?

11.2 Suppose a number *n* can be factorized as $p\times q$ where *p* and *q* are different prime numbers. List the factors of *n*. What is their sum?

11.3 The answer to 11.2 is that the factors of *pq*, where *p* and *q* are different prime numbers, are 1, *p*, *q* and *pq*, whose sum is $1+p+q+pq$. How can this expression be factorized?

11.4 From 11.3 we see that the sum of the prime factors of *pq* can be written as $(1+p)(1+q)$. We now try to generalize this. The number 66 has the prime factorization $2\times 3\times 11$. We have seen that the sum of its factors is 144. Can you now guess a formula for the sum of the factors of a number *n* which can be factorized as *pqr* where *p*, *q* and *r* are three different prime numbers. Try out your guess in the case of the number 70.

11.5 At this stage it is useful to introduce some notation to avoid the need to keep say "the sum of the factors of *n*". The standard notation for this is $\sigma(n)$. So, for example, $\sigma(22)=36$, and the general result of 11.1 can be written as $\sigma(p)=1+p$. The result of 11.3 can be written as $\sigma(pq)=(1+p)(1+q)$. In answering 11.4 you may have guessed that $\sigma(pqr)=(1+p)(1+q)(1+r)$, where *p*, *q* and *r* are three different primes. Can you now prove this?

11.6 We now look at the case of numbers which are powers of a prime number. Begin by evaluating $\sigma(4)$, $\sigma(8)$, $\sigma(9)$ and $\sigma(27)$. Can you use these results to obtain and prove a general formula for $\sigma(p^k)$, where *p* is a prime number and *k* is a positive integer?

11.7 Can you find a general formula for $\sigma(n)$ where $n$ has the prime factorization $p^a q^b r^c s^d \ldots$, where *p*, *q*, *r*, s, … are different primes.

11.8 The table shows the factorization into primes of the numbers given in Question 11, and the number of different factors each number has.

Use this information to help you guess, and then prove, a formula for the number of factors of a number *n* in terms of the factorization

$$n = p^a q^b r^c s^d \ldots$$

of *n* into different primes.

| *number* | *factorization* | *number of factors* |
|---|---|---|
| 3 | $3^1$ | 2 |
| 22 | $2^1\times 11^1$ | 4 |
| 40 | $2^3\times 5^1$ | 8 |
| 66 | $2^1\times 3^1\times 11^1$ | 8 |
| 70 | $2^1\times 5^1\times 7^1$ | 8 |

12. The sum $\qquad$ one + four = seventy
becomes correct if we replace each word by the number of letters in it to give $3+4=7$. Using the same convention, which of these words could be substituted for $x$ to make the sum

three + five = $x$ $\qquad$ true?

A eight $\qquad$ B nine $\qquad$ C twelve $\qquad$ D seventeen $\qquad$ E eighteen

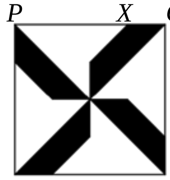
*Solution:* **D**

If we replace 'three' and 'five' by the number of letters in these words, we obtain the equation

$$5+4=x.$$

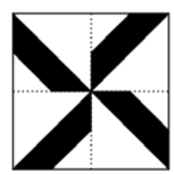
Hence $x=9$. So we need to replace $x$ by a word with 9 letters in it. Of the options we are given 'seventeen' is the only word with 9 letters in it.

13. Four congruent isosceles trapeziums are placed so that their longer parallel sides form the diagonals of a square *PQRS* as shown. The point *X* divides *PQ* in the ratio 3:1. What fraction of the square is shaded?




A $\frac{5}{16}$ $\qquad$ B $\frac{3}{8}$ $\qquad$ C $\frac{7}{16}$ $\qquad$ D $\frac{5}{12}$ $\qquad$ E $\frac{1}{2}$

*Solution:* **B**



Since *X* divides *PQ* in the ratio 3:1, it is convenient to choose units so that the side of the square has length 4, and hence *XQ* has length 1. Then the total area of the square is $4^2=16$.

If we add the dotted lines to the diagram, as shown on the right, we see that the unshaded part of the square is made up of 4 isosceles right-angled triangles in which the shorter sides have length 2, and 4 isosceles right-angled triangles in which the shorter sides have length 1. Each of the larger of these triangles has area $\frac{1}{2}(2^2)=2$, and each of the smaller triangles has area $\frac{1}{2}(1^2)=\frac{1}{2}$. Therefore the total area of the square that is not shaded is $4\times 2+4\times\frac{1}{2}=10$. Therefore the area that is shaded is $16-10=6$. Hence the fraction of the square that is shaded is $\frac{6}{16}=\frac{3}{8}$.

| | |
|---|---|
| 14. | Which of the following has the greatest value?<br>A $\left(\frac{11}{7}\right)^3$ B $\left(\frac{5}{3}\right)^3$ C $\left(\frac{7}{4}\right)^3$ D $\left(\frac{9}{5}\right)^3$ E $\left(\frac{3}{2}\right)^3$ |

*Solution:* **D**

If $x$ and $y$ are positive numbers, then $x < y$ if and only if $x^3 < y^3$. So to determine which of the given fractions when cubed results in the largest number, we need only decide which of the fractions is the largest.

To do this we use the fact that, where *p*, *q*, *r* and *s* are positive integers, we have $\frac{p}{q} < \frac{r}{s} \Leftrightarrow ps < rq$, and so we can compare the size of the fractions $\frac{p}{q}$ and $\frac{r}{s}$ by comparing the size of the integers *ps* and *qr*.

Since $11 \times 3 < 5 \times 7$, $\frac{11}{7} < \frac{5}{3}$, and since $5 \times 4 < 7 \times 3$, $\frac{5}{3} < \frac{7}{4}$. Also $3 \times 5 < 9 \times 2$ and so $\frac{3}{2} < \frac{9}{5}$. Hence the largest fraction is either $\frac{7}{4}$ or $\frac{9}{5}$. Since $7 \times 5 < 9 \times 4$, $\frac{7}{4} < \frac{9}{5}$, and therefore $\frac{9}{5}$ is the largest of the given fractions.

We deduce that $\left(\frac{9}{5}\right)^3$ is the largest of the given cubes.

| | |
|---|---|
| 15. | I have a bag of coins. In it, one third of the coins are gold, one fifth of them are silver, two sevenths are bronze and the rest are copper. My bag can hold a maximum of 200 coins. How many coins are in my bag?<br>A 101 B 105 C 153 D 195 E more information is needed |

*Solution:* **B**

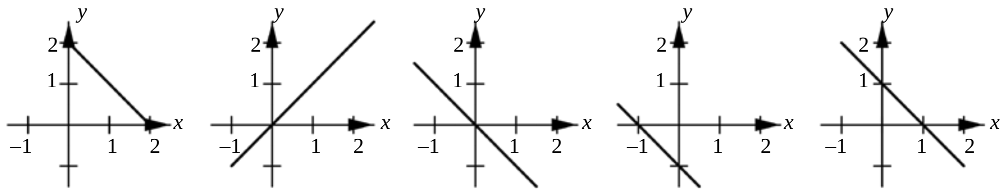
Since one third of the coins are gold, and we cannot have fractions of a coin, the number of coins in the bag must be divisible by 3. Similarly, this number must also be divisible by 5 and by 7. A number is a multiple of 3, 5 and 7 if, and only if, it is a multiple of $3 \times 5 \times 7 = 105$. Since my bag holds at most 200 coins, the only possibility is that there are 105 coins in my bag.

**Extension Problems**

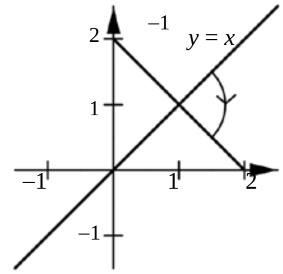
15.1 I have another bag of coins. In it, one quarter of the coins are gold, one third of them are silver, one fifth of them are bronze and the rest are copper. What is the smallest number of coins I could have in my bag?.

15.2 I have yet another bag of coins. In it, one quarter of the coins are gold, one twelfth of them are silver, three tenths are bronze and the rest are copper. What is the smallest number of coins I could have in my bag?

| | |
|---|---|
| 16. | Which diagram shows the graph $y = x$ after it has been rotated $90^\circ$ clockwise about the point (1, 1)?<br>A B C D E |



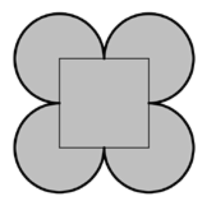

*Solution:* **A**




The line $y = x$ goes through the point (1, 1). So, if it is rotated about this point, the resulting line also goes through (1, 1). This means it could only be the graph in diagrams A or B. However, diagram B shows the graph of $y = x$. So, assuming that one of the diagrams correct, it must be diagram A.

This answer is good enough for the IMC. However, to show that A really is correct , we need an additional argument. However, it is easily seen from the geometry, that if we rotate the line $y = x$ through $90^\circ$ clockwise about the point (1, 1), we obtain the line shown in diagram A.

17. The diagram shows four equal discs and a square. Each disc touches its two neighbouring discs. Each corner of the square is positioned at the centre of a disc. The side length of the square is $2/\pi$. What is the length of the perimeter of the figure?



A 3 B 4 C $\frac{3\pi}{2}$ D 6 E $2\pi$

*Solution:* **D**

The radius of each disc is $1/\pi$, and hence its circumference is $2\pi \times 1/\pi = 2$. The perimeter of the figure is made up of three-quarters of each of the four discs. Hence the length of the perimeter is $4 \times (\frac{3}{4} \times 2) = 6$.

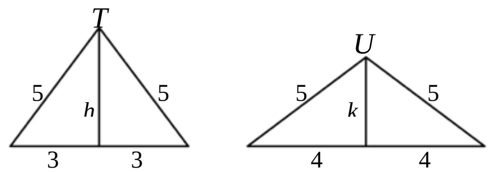
**Extension Problem**

17.1 What is the area of the figure?

18. The triangle $T$ has sides of lengths 6, 5, 5. The triangle $U$ has sides of length 8, 5, 5. What is the ratio area $T$ : area $U$ ?

A 9 : 16 B 3 : 4 C 1 : 1 D 4 : 3 E 16 : 9

*Solution:* **C**




We let the triangle $T$ have height $h$ and triangle $U$ have height $k$.
By Pythagoras' Theorem $h^2 + 3^2 = 5^2$ and $k^2 + 4^2 = 5^2$. Hence $h = 4$ and $k = 3$. It follows that both triangles are made up of two triangles with side lengths 3, 4 and 5. So their areas are the same.
[Using the formula area $= \frac{1}{2}(\text{base} \times \text{height})$, we see that $T$ has area $\frac{1}{2}(6 \times 4) = 12$, and triangle $U$ has area $\frac{1}{2}(8 \times 3) = 12$.]
Therefore, area $T$ : area $U$ = 1 : 1.

| | |
|---|---|
| 19. | Which of the expressions below is equivalent to $(x \div (y \div z)) \div ((x \div y) \div z)$? |
| | A 1     B $\dfrac{1}{xyz}$     C $x^2$     D $y^2$     E $z^2$ |

*Solution:* **E**

We have $((x \div (y \div z)) \div ((x \div y) \div z) = \dfrac{x/(y/z)}{(x/y)/z} = \dfrac{x \times \frac{z}{y}}{\frac{x}{y} \times \frac{1}{z}} = \dfrac{\frac{xz}{y}}{\frac{x}{yz}} = \dfrac{xz}{y} \times \dfrac{yz}{x} = \dfrac{xyz^2}{xy} = z^2$.

**Extension Problem**

19.1 Simplify the expression $(x \div (y \div (z \div w))) \div (((x \div y) \div z) \div w)$.

19.2 How many different expressions may be obtained from

$$x \div y \div z \div w$$

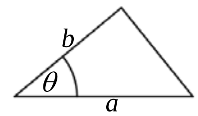
by inserting brackets?

| | |
|---|---|
| 20. | Jack's teacher asked him to draw a triangle of area 7 cm$^2$. Two sides are to be of length 6 cm and 8 cm. How many possibilities are there for the length of the third side of the triangle?<br>A 1 B 2 C 3 D 4 E more than 4 |

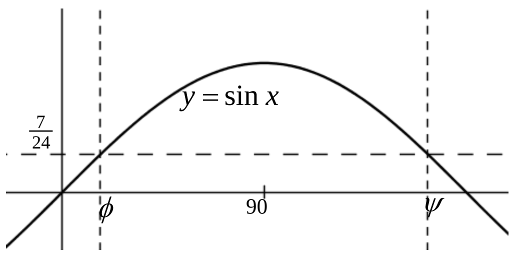
*Solution:* **B**

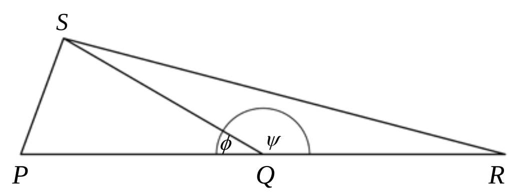
This problem could be tackled in several different ways. We give two methods here, and leave a third method to the extension problems.

*Method* 1: This is based on the fact that if a triangle has sides of lengths $a$ and $b$ and the angle between the sides is $\theta$, then the area of the triangle is given by $\frac{1}{2}ab\sin\theta$. If you are not familiar with this, ask your teacher.




Let the angle between the sides of length 8 cm and 6 cm be $\theta$. Then the area of the triangle is $\frac{1}{2}(8\times 6)\sin\theta \ = 24\sin\theta$ cm$^2$. So we need to have $24\sin\theta = 7$, and hence $\sin\theta = \frac{7}{24}$. This equation has two solutions in the range $0<\theta<180$, say $\phi$ and $\psi$ as we see from the graph. Note that $\phi+\psi=180$, and so we can represent the two resulting triangles as shown in the diagram on the right below (for clarity, this is not drawn to scale).

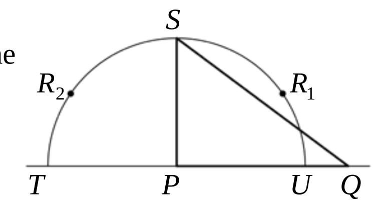



In the diagram above on the right $PQ = QR = 8$ cm and $QS = 6$ cm. So $PQS$ and $QRS$ are the two triangles with sides of lengths 6cm and 8cm, and area 7 cm$^2$. It is evident that the third sides of these triangles, $PS$ and $RS$ have different lengths. [You are encouraged to calculate their lengths in Extension Problem 20.6, below.] So there are two possibilities for the length of the third side.

*Method* 2: Here we use the formula $\frac{1}{2}(\text{base}\times\text{height})$ for the area of a triangle. We consider triangles which have a base, say $PQ$, of length 8 cm, and where the third vertex, $R$, is such that $PR$ has length 6 cm. By symmetry we need only consider triangles where $R$ lies above $PQ$.

The third vertex $R$ lies on a semicircle with centre $P$ and radius 6 cm. The area of the triangle $PQR$ is determined by the vertical height of $R$ above $PQ$. So the largest area is when $R$ is at the point $S$ vertically above $P$, and the area is then $\frac{1}{2}(8\times 6) = 24\,\text{cm}^2$. The minimum area is when the height is zero. This occurs when $R$ coincides with either the point $T$ or the point $U$ which are the endpoints of the diameter of the semicircle. In these cases the area of the triangle is 0 cm$^2$.




Since the area drops as the height drops, as $R$ moves clockwise around the semicircle from $S$ to $U$ there will be exactly one point, say $R_1$, where the area becomes 7 cm$^2$, and when $R$ moves anticlockwise from $S$ to $T$ there will be exactly one point, say $R_2$, where the area becomes 7 cm$^2$. So there are precisely two values for the length of $QR$.

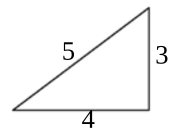
**Extension Problems**

In these problems we consider a third method based on *Heron's Formula* which gives the area of a triangle in terms of its side lengths. The problems 20.1 to 20.5 explain the formula and give some practice in using it. Then problem 20.6 asks you to use the formula to solve Question 20. Finally problem 20.7 asks you to prove that the formula is correct.

The formula is attributed to Heron of Alexander (sometimes known as Hero). Not much is known about his life, but it is generally accepted that he lived during around 50AD.

Suppose that a triangle has side lengths *a*, *b* and *c*. We let *s* be half the length of the perimeter, that is, $s = \frac{1}{2}(a+b+c)$. Often *s* is called the *semi-perimeter* of the triangle. *Heron's Formula* says that the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$ .

20.1 As an example, we first consider the right angled triangle with side lengths 3 cm, 4 cm and 5 cm. Since this triangle has base 4 cm and height 3cm, its area is $\frac{1}{2}(4 \times 3) = 6$ cm$^2$.




Calculate the values of *s*, $s-a$, $s-b$ and $s-c$ for this triangle and check that Heron's formula also gives 6 cm$^2$ for the area of the triangle.

20.2 Check that Heron's Formula gives the correct area for the right angled triangle with side lengths 5 cm, 12 cm and 13 cm.

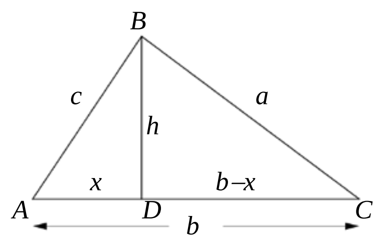
20.3 Use Heron's Formula to find the area of an equilateral triangle each of whose sides has length 1 cm.

20.4 Use Heron's Formula to find the area of a triangle with sides lengths 8 cm, 10 cm and 14 cm.

20.5 Suppose we have a triangle with sides of lengths 8 cm and 10 cm and with area 36 cm$^2$. We use Heron's Formula to find the possible lengths of the third side. Because Heron's Formula involves the semi-perimeter of the triangle, it is convenient to suppose that the third side has length 2*x* cm. Then the semi-perimeter of the triangle is $\frac{1}{2}(8+10+2x)$ $= 9 + x$. Heron's Formula gives $\sqrt{(9+x)(1+x)(-1+x)(9-x)} = 36$. That is, $\sqrt{(9^2 - x^2)(-1+x^2)} = 36$. Show that if we let $y = x^2$, this gives $y^2 - 82y + 1377 = 0$. Hence find the possible lengths of the third side of the triangle.

20.6 Use Heron's Formula to find the possible lengths of a triangle which has sides of lengths of 6cm and 8cm and area 7 cm$^2$, as in Question 20.

20.7 We now indicate a proof of Heron's Formula. Suppose that we have a triangle *ABC* with side lengths *a*, *b* and *c*. The triangle must have at least two angles which are less than $90^\circ$. So we can assume that $\angle BAC$ and $\angle BCA$ are both less than $90^\circ$. Let *D* be the point where the perpendicular from *B* to *AC* meets *AC*. Suppose that *BD* has length *h* and *AD* has length *x*.

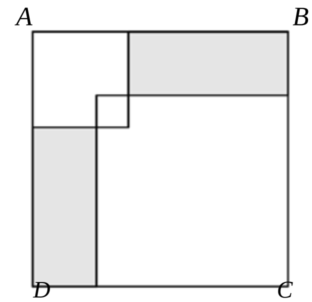



Pythagoras' Theorem applied to the right angled triangles *ADB* and *CDB* now gives $x^2 + h^2 = c^2$ and $(b-x)^2 + h^2 = a^2$. Show that it follows from these equations that $x = (c^2 - a^2 - b^2)/2b$ and hence obtain a formulas for *h* and the area of the triangle in terms of *a*, *b* and *c*.

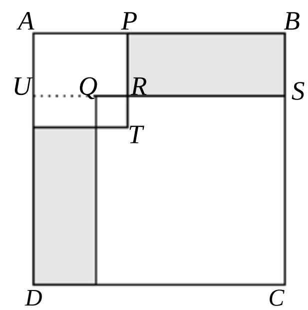
21. The square $ABCD$ has an area of 196. It contains two overlapping squares; the larger of these squares has an area 4 times that of the smaller and the area of their overlap is 1. What is the total area of the shaded region?

A 44 B 72 C 80 D 152

E more information is needed.




*Solution:* **B**




We let $P$, $Q$, $R$, $S$, $T$ and $U$ be the points as labelled in the diagram. Since the square $ABCD$ area 196, it has side length 14. We suppose that the smaller square inside $ABCD$ has side length $x$. Since the larger square inside $ABCD$ has area 4 times that of the smaller square it will have side length $2x$.

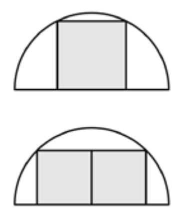
We note that the diagram is symmetric about the line $AC$. Hence the overlap of the squares is itself a square. Since this square has area 1, then it has side length 1. So $QR$ has length 1. As $UR = AP$, $UR$ has length $x$. Now $QS$ has length $2x$. Therefore $US$ has length $x + 2x - 1 = 3x - 1$. The length of $US$ is the same as the side length of the square $ABCD$. Therefore $3x - 1 = 14$. Hence $x = 5$.

So the larger square inside $ABCD$ has side length 10 and hence area 100, and the smaller square has area 25. As their overlap has area 1, the total area they cover is $100 + 25 - 1 = 124$. Therefore, the shaded area is $196 - 124 = 72$.

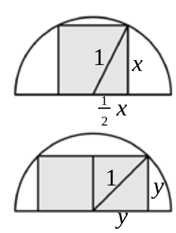
22. The diagrams show squares placed inside two identical semicircles In the lower diagram the two squares are identical.

What is the ratio of the areas of the two shaded regions?

A 1: 2 B 2 : 3 C 3 : 4 D 4 : 5 E 5 : 6



*Solution:* **D**




We choose units so that the two semicircles have radius 1. We let $x$ be the side length of the square in the upper diagram, and $y$ be the side lengths of the squares in the lower diagram.

By Pythagoras' Theorem, we have $x^2 + (\frac{1}{2}x)^2 = 1$ and $y^2 + y^2 = 1$. That is $\frac{5}{4}x^2 = 1$ and $2y^2 = 1$. Therefore $x^2 = \frac{4}{5}$ and $y^2 = \frac{1}{2}$. The shaded region in the top diagram consists of a square of side length $x$ and hence of area $x^2$, that is, $\frac{4}{5}$. The shaded region in the lower diagram consists of two squares each of side length $y$, and hence it has area $2y^2$, that is, 1. Hence the ratio of the two shaded regions is $\frac{4}{5}:1 = 4:5$.

23. Four brothers are discussing the order in which they were born. Two are lying and two are telling the truth. Which two are telling the truth?

Alfred: “Bernard is the youngest.”

Horatio: “Bernard is the oldest and I am the youngest.”

Inigo: “I was born last.”

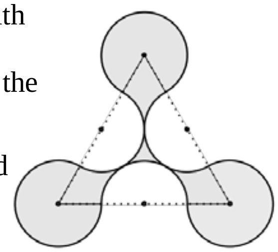
Bernard: “I’m neither the youngest nor the oldest.”

A Bernard and Inigo B Horatio and Bernard C Alfred and Horatio D Alfred and Bernard E Inigo and Horatio

*Solution:* **A**

What Horatio says contradicts each of the statements made by his brothers. So he cannot be one of the two who are telling the truth. What Alfred says contradicts what both Inigo says and what Bernard says. So the two who are telling the truth cannot include Alfred. We deduce that the two brother who are telling the truth are Bernard and Inigo. This is indeed possible. For example, if Alfred was born first, Horatio second, Bernard third and Inigo last, then just Bernard and Inigo are telling the truth.

24. The diagram shows a shaded shape bounded by circular arcs with the same radius. The centres of three arcs are the vertices of an equilateral triangle; the other three centres are the midpoints of the sides of the triangle. The sides of the triangle have length 2.



What is the difference between the area of the shaded shape and the area of the triangle?

A $\frac{\pi}{6}$ B $\frac{\pi}{4}$ C $\frac{\pi}{3}$ D $\frac{\pi}{2}$ E $\pi$

*Solution:* **B**

Since the sides of the triangle have length 2, each of the circular arcs has radius $\frac{1}{2}$. The difference between the area of the shaded shape and that of the triangle, is the difference between the area of the 3 shaded sectors of circles outside the triangle, and that of the 3 unshaded semicircles within the triangle.

The angles of the equilateral triangle are $60^o$ which is $\frac{1}{6}$th of a complete revolution. So the areas of the shaded sectors of circles outside the triangle are $\frac{5}{6}$ths of the total areas of these circles, each of which has radius $\frac{1}{2}$. So the area of these sectors is $3\times\left(\frac{5}{6}(\pi(\frac{1}{2})^2\right)=\frac{5}{8}\pi$. The area of the 3 unshaded semicircles inside the triangle is $3\times\left(\frac{1}{2}(\pi(\frac{1}{2})^2\right)=\frac{3}{8}\pi$. The difference between these is $\frac{5}{8}\pi-\frac{3}{8}\pi=\frac{1}{4}\pi$. Hence, this is the difference between the area of the shaded shape and the area of the triangle.

25. In 1984 the engineer and prolific prime-finder Harvey Dubner found the biggest known prime each of whose digits is either a one or a zero. The prime can be expressed as $\frac{10^{641}\times(10^{640}-1)}{9}+1$. How many digits does this prime have?

A 640 B 641 C 1280 D 1281 E $640\times 641$

*Solution:* **D**

The number $10^{640}-1$ consists of a string of 640 nines when written out in standard form. So the number $10^{641}\times(10^{640}-1)$ consists of a string of 640 nines followed by 641 zeros. Therefore $\frac{10^{641}\times(10^{640}-1)}{9}$ consists of a string of 640 ones followed by 641 zeros. Adding 1 just changes the final digit from zero to a one. So the prime is

$$\overbrace{111...111}^{640}\overbrace{000...000}^{640}1$$

and therefore we see that it has $640+640+1=1281$ digits.

**Extension Problems**

The number found by Harvey Dubner is so large that you need a clever algorithm and a very fast computer if you wish to find out whether or not it is a prime number. In these problems we invite you to consider the easier problem of testing whether numbers of this kind are divisible by some small prime numbers.

It is useful to introduce some notation. We use $D_n$ to stand for the number which when written in standard form consists of *n* ones followed by *n* zeros and then a single 1 as the units digit. So $D_1=101$, $D_2=11001$, $D_3=1110001$, $D_4=111100001$ and so on. Using this notation, we could write the number that Question 25 asks you about as $D_{640}$.

25.1 Explain why none of the numbers in the infinite sequence $D_1$, $D_2$, $D_3$, … is divisible by either 2 or by 5.

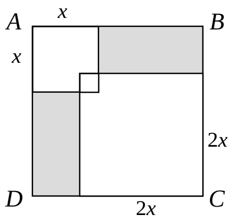
25.2 Determine which of the numbers $D_1$, $D_2$, $D_3$, $D_4$, $D_5$ and $D_6$ is divisible by 3.

25.3 Generalize your answer to 25.2. For which values of *n* is the number $D_n$ a multiple of 3? Prove that your answer is correct. [You could tackle this directly. Alternatively, you could use the test for divisibility by 3 given in Extension Problem 1.1.]

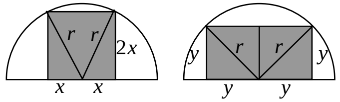
25.4 Determine for which values of *n* the number $D_n$ is divisible by 11 [There is a test in terms of its digits for a number to be divisible by 11 which you could use here if you know it or your teacher tells you it, or you could tackle the problem directly.]

25.5 Determine for which values of *n* the number $D_n$ is divisible by 7. [The test in terms of its digits for a number to be divisible by 7 is rather more complicated, and it is better to consider this case directly. That is why we have asked you to consider divisibility by 11 first.]

**21. B** The large square has area $196 = 14^2$. So it has side-length 14. The ratio of the areas of the inner squares is 4 : 1, so the ratio of their side-lengths is 2 : 1. Let the side-length of the larger inner square be $2x$, so that of the smaller is $x$. The figure is symmetric about the diagonal $AC$ and so the overlap of the two inner squares is also a square which therefore has side-length 1. Thus the vertical height can be written as $x + 2x - 1$. Hence $3x - 1 = 14$ and so $x = 5$. Also, the two shaded rectangles both have side-lengths $2x - 1$ and $x - 1$; that is 9 and 4. So the total shaded area is 72.

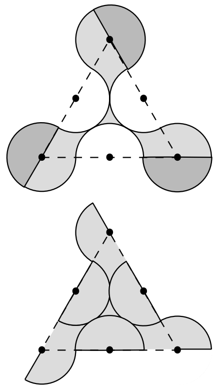



**22. D** Let the radius of each semicircle be $r$. In the left-hand diagram, let the side-length of the square be $2x$. By Pythagoras' Theorem, $x^2 + (2x)^2 = r^2$ and so $5x^2 = r^2$. So this shaded area is $4x^2 = \frac{4r^2}{5}$. In the right-hand diagram, let the side-length of each square be $y$. Then by Pythagoras' Theorem, $y^2 + y^2 = r^2$ and so this shaded area is $r^2$. Therefore the ratio of the two shaded areas is $\frac{4}{5} : 1 = 4 : 5$.




**23. A** If Alfred is telling the truth, the other three are lying (as their statements would then be false) and we know this is not the case. Hence Alfred is lying. Similarly, if Horatio is telling the truth, the other three are lying which again cannot be the case. So Horatio is lying. Hence the two who are telling the truth are Bernard and Inigo. (A case where this situation would be realised would be if the brothers in descending order of age were Alfred , Bernard, Horatio and Inigo.)

**24. B** The length of the side of the triangle is equal to four times the radius of the arcs. So the arcs have radius $2 \div 4 = \frac{1}{2}$. In the first diagram, three semicircles have been shaded dark grey. The second diagram shows how these semicircles may be placed inside the triangle so that the whole triangle is shaded. Therefore the difference between the area of the shaded shape and the area of the triangle is the sum of the areas of three sectors of a circle. The interior angle of an equilateral triangle is 60°, so the angle at the centre of each sector is $180° - 60° = 120°$. Therefore each sector is equal in area to one-third of the area of a circle. Their combined area is equal to the area of a circle of radius $\frac{1}{2}$. So the required area is $\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.



**25. D** $\left(10^{640} - 1\right)$ is a 640-digit number consisting entirely of nines. So $\frac{\left(10^{640} - 1\right)}{9}$ is a 640-digit number consisting entirely of ones.
Therefore $\frac{10^{641} \times \left(10^{640} - 1\right)}{9}$ consists of 640 ones followed by 641 zeros. So $\frac{10^{641} \times \left(10^{640} - 1\right)}{9} + 1$ consists of 640 ones followed by 640 zeros followed by a single one. Therefore it has 1281 digits.

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# UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 7TH FEBRUARY 2013


Organised by the **United Kingdom Mathematics Trust**
**from the School of Mathematics, University of Leeds**

*http://www.ukmt.org.uk*


## SOLUTIONS LEAFLET

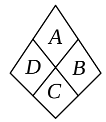
This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates. More comprehensive solutions are on the website.

**The UKMT is a registered charity**

**1. D** In order to be a multiple of 6, a number must be both even and a multiple of 3. Of the numbers given, only B 999 998 and D 999 996 are even. Using the rule for division by 3, we see that, of these two, only 999 996 is a multiple of 3.

**2. B** 180 000 eggs per hour is equivalent to 3000 eggs per minute, i.e. to 50 eggs per second.

**3. E** The figure is itself a quadrilateral. It can be divided into four small quadrilaterals labelled A, B, C, D. There are also four quadrilaterals formed in each case by joining together two of the smaller quadrilaterals: A and B; B and C; C and D; D and A.




**4. D** The number of seeds in a special packet is $1.25 \times 40 = 50$. So the number of seeds which germinate is $0.7 \times 50 = 35$.

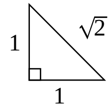
**5. E** A wheatear travels the distance of almost 15 000 km in approximately 50 days. This is on average roughly 300 km per day.

**6. E** In order, the values of the expressions given are: $1 - 0 = 1$; $2 - 1 = 1$; $9 - 8 = 1$; $64 - 81 = -17$; $625 - 1024 = -399$.

7. **A** Only two colours are needed for the upper four faces of the octahedron. If, for example, blue and red are used then these four faces may be painted alternately red and blue. Consider now the lower four faces: every face adjacent to an upper blue face may be painted red and every face adjacent to an upper red face may be painted blue. So only two colours are required for the whole octahedron.

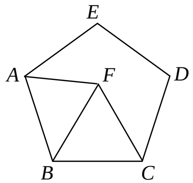
8. **D** Let the number of scores of 1 be $n$. Then the product of the scores is $1^n \times 2 \times 3 \times 5 = 30$. Therefore $1 \times n + 2 + 3 + 5 = 30$, i.e. $n = 20$. So Jim threw 23 dice.

9. **A** Let the length of the shorter sides of the cards be 1 unit. Then, by Pythagoras' Theorem, the length of the hypotenuse of each card is $\sqrt{1^2 + 1^2} = \sqrt{2}$.




So the lengths of the perimeters of the five figures in order are: $4\sqrt{2}$; $4 + 2\sqrt{2}$; $4 + 2\sqrt{2}$; $6$; $4 + 2\sqrt{2}$. Also, as $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4} > 2$ we see that $\frac{3}{2} > \sqrt{2}$. Therefore, $4\sqrt{2} < 6 < 4 + 2\sqrt{2}$. So figure A has the shortest perimeter.

10. **C** The sum of the interior angles of a pentagon is 540° so $\angle ABC = 540° \div 5 = 108°$. Each interior angle of an equilateral triangle is 60°, so $\angle FBC = 60°$.

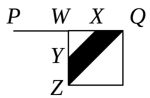



Therefore $\angle ABF = 108° - 60° = 48°$. As $ABCDE$ is a regular pentagon, $BC = AB$. However, $BC = FB$ since triangle $BFC$ is equilateral.
So triangle $ABF$ is isosceles with $FB = AB$.
Therefore $\angle FAB = \angle AFB = (180° - 48°) \div 2 = 66°$.

11. **C** We first look at $66 = 2 \times 3 \times 11$. Its factors involve none, one, two or all three of these primes. So the factors are 1, 2, 3, 11, 6, 22, 33, 66; and their sum is $144 = 12^2$. Similarly, we can check that the sum of the factors of 3, 22, 40 and 70 is, respectively, $4 = 2^2$, $36 = 6^2$, 90 and $144 = 12^2$. So 40 is the only alternative for which the sum of the factors is not a square number.

12. **D** As the words 'three' and 'five' contain 5 and 4 letters respectively, their 'sum' will be a 9-letter word. Of the alternatives given, only 'seventeen' contains 9 letters.

13. **B** The diagram shows the top-right-hand portion of the square. The shaded trapezium is labelled $QXYZ$ and $W$ is the point at which $ZY$ produced meets $PQ$.



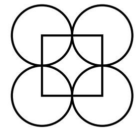

As $QXYZ$ is an isosceles trapezium, $\angle QZY = \angle ZQX = 45°$.
Also, as $YX$ is parallel to $ZQ$, $\angle XYW = \angle WXY = 45°$. So $WYX$ and $WZQ$ are both isosceles right-angled triangles. As $\angle ZWQ = 90°$ and $Z$ is at the centre of square $PQRS$, we deduce that $W$ is the midpoint of $PQ$. Hence $WX = XQ = \frac{1}{4}PQ$. So the ratio of the side-lengths of similar triangles $WYX$ and $WZQ$ is 1 : 2 and hence the ratio of their areas is 1 : 4.
Therefore the area of trapezium $QXYZ = \frac{3}{4} \times$ area of triangle $ZWQ = \frac{3}{32} \times$ area $PQRS$ since triangle $ZWQ$ is one-eighth of $PQRS$. So the fraction of the square which is shaded is $4 \times \frac{3}{32} = \frac{3}{8}$.

14. **D** As all the fractions are raised to the power 3, the expression which has the largest value is that with the largest fraction in the brackets.
Each of these fractions is a little larger than $1\frac{1}{2}$. Subtracting $1\frac{1}{2}$ from each in turn, we get the fractions $\frac{1}{14}, \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, 0$, the largest of which is $\frac{3}{10}$ (because $0 < \frac{1}{14} < \frac{1}{6} < \frac{1}{4} = \frac{2\frac{1}{2}}{10} < \frac{3}{10}$). Hence $\left(\frac{9}{5}\right)^3$ is the largest.

15. **B** From the information given, we may deduce that the number of coins is a multiple of each of 3, 5, 7. Since these are distinct primes, their lowest common multiple is $3 \times 5 \times 7 = 105$. So the number of coins in the bag is a multiple of 105. So there are 105 coins in the bag since 105 is the only positive multiple of 105 less than or equal to 200.

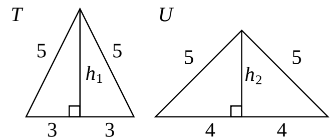
16. **A** The image of a straight line under a rotation is also a straight line. The centre of rotation, the point (1, 1), lies on the given line and so also lies on the image. The given line has slope 1 and so its image will have slope −1. Hence graph A shows the image.

17. **D** The radius of each disc in the figure is equal to half the side-length of the square, i.e. $\frac{1}{\pi}$. Because the corners of a square are right-angled, the square hides exactly one quarter of each disc.



So three-quarters of the perimeter of each disc lies on the perimeter of the figure. Therefore the length of the perimeter is $4 \times \frac{3}{4} \times 2\pi \times \frac{1}{\pi} = 6$.

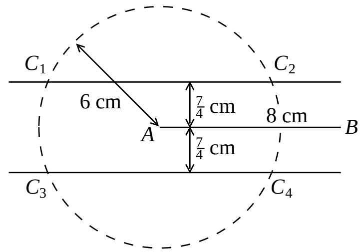
18. **C** The diagrams show isosceles triangles $T$ and $U$. The perpendicular from the top vertex to the base divides an isosceles triangle into two congruent right-angled triangles as shown in both $T$ and $U$.




Evidently, by Pythagoras' Theorem, $h_1 = 4$ and $h_2 = 3$. So both triangles $T$ and $U$ consist of two '3, 4, 5' triangles and therefore have equal areas.

19. **E** $(x \div (y \div z)) \div ((x \div y) \div z) = \left(x \div \frac{y}{z}\right) \div \left(\left(\frac{x}{y}\right) \div z\right) = \left(x \times \frac{z}{y}\right) \div \left(\frac{x}{y} \times \frac{1}{z}\right)$
$= \frac{xz}{y} \div \frac{x}{yz} = \frac{xz}{y} \times \frac{yz}{x} = z^2$.

20. **B** Let the base $AB$ of the triangle be the side of length 8 cm and let $AC$ be the side of length 6 cm. So $C$ must lie on the circle with centre $A$ and radius 6 cm as shown. The area of the triangle is to be 7 cm$^2$, so the perpendicular from $C$ to $AB$ (or to $BA$ produced) must be of length $\frac{7}{4}$ cm.




The diagram shows the four possible positions of $C$. However, since $\angle BAC_1 = \angle BAC_3$ and $\angle BAC_2 = \angle BAC_4$, these correspond to exactly two possibilities for the length of the third side $AC$. The diagrams below show the two possibilities.